MOTION OF A GYROSCOPE IN UNIVERSAL SUSPENSION ON A UNIFORMLY ROTATING BASE

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Kh. L. SMOLITSKII
(Leningrad)
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Knowledge of the integral of the equations of motion of a gyroscope enables us to investigate certain details of the gyroscope axis trajectory. Various approximations of this trajectory are discussed.

1. Let a gyroscope be mounted in universal suspension on a base rotating uniformly with the angular velocity $\omega > 0$. The axis of the outer frame forms the angle $\pi/2 - \varphi$ with the axis of rotation of the base. We assume that the gravitational force (its acceleration g) is directed along the axis of the outer frame, while the center of gravity of the gyroblock is shifted relative to the axis of rotation by the quantity $l = \omega\lambda$ proportional to the angular velocity of the base. We shall be concerned principally with the case $\lambda = 0$, when the gyroscope is in equilibrium.

The position of the frames is given in terms of the angles α, the angle of rotation of the outer frame relative to the base ($\alpha = 0$ when the axes of the gyroscope, outer frame, and rotation of the base lie in the same plane), and β, the angle of rotation of the inner frame relative to the outer ($\beta = 0$ when the frame planes are perpendicular).

Let I, I_0 be the axial and equatorial moments of inertia of the rotor; I_{1x}, I_{1y}, I_{1z} are the moments of inertia of the inner frame relative to the rotor axis, to the axis of the inner frame, and to the axis perpendicular to the first two axes; I_{2x}, I_{2y}, I_{2z} are the moments of inertia of the outer frame relative to the axis perpendicular to the frame plane, to the inner frame axis, and to the outer frame axis; m, Ω are the mass and total angular velocity of the rotor. Let us introduce the dimensionless quantities

$$P = \frac{I_0 + I_{1y}}{I}, \qquad Q = \frac{I_0 + I_{1z} + I_{2z}}{I}, \qquad R = \frac{I_0 + I_{1z} - I_{1x}}{I}$$

$$S = \frac{(I_{1z} - I_{1y}) + (I_{2x} - I_{2y})}{I}, \qquad a = \frac{mgl}{I\Omega\omega} = \frac{mg\lambda}{I\Omega}, \qquad \varkappa = a - \sin\varphi$$

Let us denote by γ the angle between the axis of rotation and the gyroscope axis; we have $\cos\gamma = \sin\varphi\sin\beta + \cos\varphi\cos\alpha\cos\beta$. We introduce the functions

$$\Phi(\alpha, \beta) = a\sin\beta - \cos\gamma = \varkappa\sin\beta - \cos\varphi\cos\alpha\cos\beta$$

$$\Psi(\alpha, \beta) = {}^1/_2\,[R\cos^2\gamma + S\cos^2\varphi\sin^2\alpha],$$

$$p(\alpha, \beta) = (R - P)\cos\varphi\cos\alpha - 2R\cos\beta\cos\gamma$$

$$f(\beta) = Q - R\sin^2\beta \qquad (f(\beta) > 0)$$

From now on differentiation with resect to time t will be denoted by a dot; differentiation with resect to α and β will be denoted by a prime. In this notation the equations of motion of the gyroscope can be written as

$$f(\beta)\,\alpha^{\cdot\cdot} + f'(\beta)\,\alpha^{\cdot}\beta^{\cdot} + [\Omega\cos\beta + \omega p(\alpha, \beta)]\,\beta^{\cdot} + \omega\,[\Omega\Phi + \omega\Psi]_\alpha' = 0 \qquad (1.1)$$

$$P\beta'' - {}^1/_2 f'(\beta)\alpha'^2 - [\Omega\cos\beta + \omega p(\alpha,\beta)]\alpha' + \omega[\Omega\Phi + \omega\Psi]_\beta' = 0 \qquad (1.2)$$

We shall investigate system (1.1),(1.2) under the initial conditions

$$\alpha|_{t=0} = \alpha_0, \quad \beta|_{t=0} = \beta_0, \quad \alpha'|_{t=0} = \omega\chi_1, \quad \beta'|_{t=0} = \omega\chi_2 \qquad (1.3)$$

where $\alpha_0, \beta_0, \chi_1, \chi_2$ are arbitrary numbers ($|\beta_0| < {}^1/_2\pi$). System (1.1),(1.2) has a first integral which can be expressed in abbreviated form with the aid of the function $F(\alpha,\beta) = \Omega\Phi + \omega\Psi$. We have

$$f(\beta)\alpha'^2 + P\beta'^2 = 2\omega[F_0 - F(\alpha,\beta)] + \omega^2[f_0\chi_1^2 + P\chi_2^2] \qquad (1.4)$$
$$f_0 = f(\beta_0), \quad F_0 = F(\alpha_0, \beta_0)$$

By virtue of nonnegativeness of the left-hand side of (1.4) and the assumed positiveness of ω, the gyroscope moves in such a way that the point (α, β) of the plane $\alpha\beta$ remains in the domain G_ω, defined by the inequality $F_0 - F(\alpha,\beta) + {}^1/_2\omega[f_0\chi_1^2 + P\chi_2^2] \geqslant 0$. This domain will henceforth be referred to as the "potential well". We shall always assume that $\omega \ll |\Omega|$. Because of this the potential well differs little from the domain G_0 defined by the inequality $\Omega(\Phi_0 - \Phi(\alpha,\beta)) \geqslant 0$, $(\Phi_0 = \Phi(\alpha_0, \beta_0))$.

Since the functions $\Phi(\alpha,\beta)$ and $\psi(\alpha,\beta)$ are 2π-periodic and even in α, it follows that the domains G_ω and G_0 are symmetrical with respect to the staight lines $\alpha = k\pi$ (k is an integer).

We note that if the sign of Ω changes to its opposite, then the domain G_0 is replaced by the complementary portion of the plane $\alpha\beta$. On the other hand, it is evident that the direction of rotation of the rotor has little effect on the motion of the gyroscope axis. Taken together, these two facts indicate that the gyroscope moves in such a way that the point (α, β) lies near the boundary of the domain G_0, and therefore near the edge of the potential well.

Let us convert from the independent variable t to the new variable τ in Eqs. (1.1) and (1.2) by setting $t = \omega\tau$. We introduce the small parameter $\mu = \omega/\Omega$. Problem (1.1) to (1.3) then becomes

$$\left(\cos\beta\frac{d\beta}{d\tau} + \Phi_\alpha'\right) + \mu\left[f(\beta)\frac{d^2\alpha}{d\tau^2} + f'(\beta)\frac{d\alpha}{d\tau}\frac{d\beta}{d\tau} + p\frac{d\beta}{d\tau} + \Psi_\alpha'\right] = 0 \qquad (1.5)$$

$$\left(-\cos\beta\frac{d\alpha}{d\tau} + \Phi_\beta'\right) + \mu\left[P\frac{d^2\beta}{d\tau^2} - \frac{1}{2}f'(\beta)\left(\frac{d\alpha}{d\tau}\right)^2 - p\frac{d\alpha}{d\tau} + \Psi_\beta'\right] = 0 \qquad (1.6)$$

$$\alpha|_{\tau=0} = \alpha_0, \quad \beta|_{\tau=0} = \beta_0; \quad \left.\frac{d\alpha}{d\tau}\right|_{\tau=0} = \chi_1, \quad \left.\frac{d\beta}{d\tau}\right|_{\tau=0} = \chi_2 \qquad (1.7)$$

and instead of integral (1.4) we have Eq.

$$\Phi(\alpha,\beta) - \Phi_0 = \mu\left\{\Psi_0 - \Psi(\alpha,\beta) + \frac{1}{2}[f_0\chi_1^2 + P\chi_2^2] - \frac{1}{2}\left[f(\beta)\left(\frac{d\alpha}{d\tau}\right)^2 + P\left(\frac{d\beta}{d\tau}\right)\right]\right\} \qquad (1.8)$$

The higher-order terms of system (1.5),(1.6) contain the small parameter μ. Setting $\mu = 0$ and retaining only the first two conditions in (1.7), we arrive at the following problem for determining the limiting trajectory:

$$\frac{d\alpha}{d\tau} = \frac{\Phi_\beta'}{\cos\beta}, \quad \frac{d\beta}{d\tau} = -\frac{\Phi_\alpha'}{\cos\beta}, \quad \alpha|_{\tau=0} = \alpha_0, \quad \beta|_{\tau=0} = \beta_0 \qquad (1.9)$$

From Eqs. (1.9) we infer that $\Phi_\alpha'\alpha' + \Phi_\beta'\beta' = 0$, i.e. that the line $\Phi(\alpha,\beta) = \Phi_0$ contains a limiting trajectory. Problem (1.9) is formulated differently and solved in [1] for the case $a = 0$ and in [2 and 3] for $a \neq 0$. This trajectory will be called "kinematic", since it contains no information about the moments of inertia of the gyroscope frames or about the angular velocity of the gyroscope. In the case $a = 0$ the kinematic trajectory has the following meaning: at the initial instant the gyroscope axis establishes a certain direction in inertial space corresponding to the given α_0 and β_0 ; during subsequent rotation of the base the frames rotate in such a way that the gyroscope axis maintains a constant direction in inertial space.

Let us denote by χ_1^* and χ_2^* the values of the right-hand sides of Eqs. (1.9) at the point (α_0, β_0). The motion of the gyroscope under initial conditions (1.3) for $\chi_1 = \chi_1^*$, $\chi_2 = \chi_2^*$ will be called "motion with a compensating initial thrust". In the case $a = 0$ with a compensating initial thrust the gyroscope axis at the initial instant has zero velocity relative to inertial space. This is precisely the case considered in [1], where the linematic trajectory is said to be highly precise. In the case $a = 0$ for the linematic trajectory we have $\gamma = \gamma_0$.

For the points of the real trajectory the left-hand side of Eq. (1.8) can be written as $\cos\gamma_0 - \cos\gamma$, which is smaller in absolute value than $|\gamma - \gamma_0|$. Hence, the absolute value of the right-hand side of (1.8) can be considered a lower estimate for the error of gyroscope operation at any instant τ , provided $\alpha(\tau)$, $\beta(\tau)$, $d\alpha / d\tau$, $d\beta / d\tau$ are known. Assuming that for a real trajectory with a compensating thrust the values of the latter quantities differ from those for the kinematic trajectory by amounts not exceeding $C\mu$ (C = const), we can neglect the quantities of order μ^2 to obtain

$$\cos\gamma_0 - \cos\gamma \approx \frac{\mu}{2}\left\{S\cos^2\varphi(\sin^2\alpha_0 - \sin^2\alpha) + \left[\frac{Q-R}{\cos^2\beta_0} - (P-R)\right](\sin\varphi\cos\beta_0 - \right.$$

$$\left. - \cos\varphi\cos\alpha_0\sin\beta_0)^2 - \left[\frac{Q-R}{\cos^2\beta} - (P-R)\right](\sin\varphi\cos\beta - \cos\varphi\cos\alpha\sin\beta)^2\right\} \quad (1.10)$$

Here (α, β) is a point of the kinematic trajectory

$$\sin\varphi\sin\beta + \cos\varphi\cos\alpha\cos\beta = \cos\gamma_0$$

The right-hand side vanishes identically for inertialess frames ($P = Q = R$, $S = 0$), as well as at the instants $\tau = 0$ and $\tau = 2\pi$ in the case of heavy frames. Formula (1.10) will be discussed from a different standpoint below.

If the inequality $\quad \Psi_* - \Psi(\alpha^\circ, \beta^\circ) + {}^1/_2 [f_c\chi_1^2 + P\chi_2^2] < 0 \qquad (1.11)$

is fulfilled at some point $(\alpha^\circ, \beta^\circ)$ of the kinematic trajectory $\Phi(\alpha, \beta) = \Phi_*$ then the right-hand side of (1.4) is negative at this point, so that the point $(\alpha^\circ, \beta^\circ)$ does not lie in the potential well; hence, it cannot belong to the true trajectory. Let us cite some examples. The motion for $\chi_1 = \chi_2 = 0$ will be called the "motion of a freely idling gyroscope". In the case $a = 0$ at the points of the kinematic trajectory we have Eq. $2(\Psi_* - \Psi(\alpha, \beta)) = S\cos^2\varphi(\sin^2\alpha_0 - \sin^2\alpha)$. This implies that for a freely idling gyroscope with an undisplaced gyroblock inequality (1.11) is satisfied at almost all points of the kinematic trajectory in the following cases:

$$\text{(a)}\ S > 0, \quad \alpha_* = 0; \qquad \text{(b)}\ S < 0, \qquad \alpha_* = {}^1/_2\pi$$

Writing $\eta = \operatorname{sign}\Omega$, we find that

$$\eta(\cos\gamma - \cos\gamma_*) \geqslant {}^1/_2|\mu|S\cos^2\varphi\sin^2\alpha \quad \text{in case (a)}$$

$$\eta(\cos\gamma - \cos\gamma_*) \geqslant {}^1/_2|\mu S|\cos^2\varphi\cos^2\alpha \quad \text{in case (b)}$$

These formulas give a reliable lower bound for the error of a freely idling gyroscope. In the case $0 < \gamma_0 < \pi/2$ we infer that the gyroscope axis approaches the axis of rotation of the base for $\eta = +1$ and moves away from the latter for $\eta = -1$.

If $S \neq 0$ and $\alpha_0 \neq 0$, $\alpha_0 = \frac{1}{2}\pi$, there exist portions of the kinematic trajectory which do not lie in the potential well, so that similar estimates apply.

The line $F(\alpha, \beta) = F_0$ always lies in the potential well and passes through the point (α_0, β_0). It can serve as a close approximation of the true trajectory if the "schedule" of motion is specified on it, and will be called the "potential trajectory".

In order to specify the "schedule" of motion on this line we can write out differential equations of the (1.9) type, replacing the numerators by $(\Phi_\beta' + \mu\Psi_\beta')$ and $(\Phi_\alpha' + \mu\Psi_\alpha')$ respectively, and the denominator by a quantity close to $\cos\beta$. The latter can be done in various ways and on the basis of various considerations. If we require that the differential equations become Eq. (1.9) for $\alpha = 0$ and for inertialess frames ($P = Q = R$, $S = 0$), then it is sufficient to choose the denominator equal to $\cos\beta + \frac{1}{2}\mu p(\alpha, \beta)$, even though the form of Eqs. (1.1) and (1.2) indicates that the "simplified" equations [4] are those in which the denominator is equal to $\cos\beta + \mu p(\alpha, \beta)$

2. Kinematic trajectories are investigated in [1] for $\alpha = 0$ and in [2 and 3] for $\alpha \neq 0$. Hence, we shall merely describe them briefly here. Kinematic trajectories are the datum levels of the function $\Phi(\alpha, \beta)$. Limiting ourselves to the domain $|\beta| \leq \frac{1}{2}\pi$, $-\frac{1}{2}\pi < \alpha \leq \pi$ and assuming that $\cos\varphi > 0$, from Eqs. $\Phi_\alpha' = \Phi_\beta' = 0$ we obtain the following fixed points of the function $\Phi(\alpha, \beta)$:

1) $\alpha = 0, \quad \beta = -\operatorname{arc\,tg}\dfrac{\varkappa}{\cos\varphi}; \quad \Phi = -\Phi^\circ = -\sqrt{\varkappa^2 + \cos^2\varphi}$ (the minimum point)

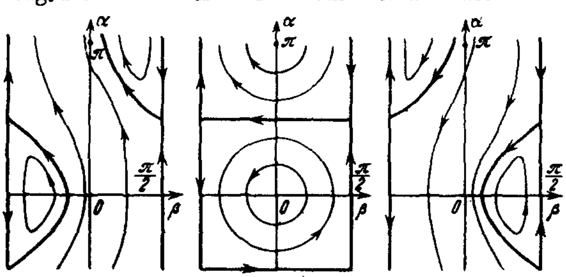
2) $\alpha = \pi, \quad \beta = \operatorname{arc\,tg}\dfrac{\varkappa}{\cos\varphi}; \quad \Phi = \Phi^\circ = \sqrt{\varkappa^2 + \cos^2\varphi}$ (the maximum point)

3) $\alpha = {}^1/_2\,\pi, \; \beta = \pm\,{}^1/_2\,\pi; \; \Phi = \pm\varkappa$ (the saddle point)

Fig. 1 shows the lines $\Phi = \text{const}$ for the cases $\varkappa > 0$, $\varkappa = 0$ and $\varkappa < 0$. In the case $\varkappa = 0$ these lines are closed; for $\varkappa \neq 0$ some of the lines are open; specifically, the line

$$\Phi(\alpha, \beta) = \Phi(\alpha_0, \beta_0)$$

is open if $|\Phi(\alpha_0, \beta_0)| < |\varkappa|$, and closed if $|\varkappa| \leqslant |\Phi \times (\alpha_0, \beta_0)| < \Phi^\circ$. The arrowheads on the lines in Fig. 1 indicate the direction of increasing T. Eq. (1.9) can be integrated in finite form. In fact let (α_1, β_1), (α_2, β_2) be two points of the kinematic trajectory $\Phi(\alpha, \beta) = \Phi(\alpha_0, \beta_0) = \Phi_0$, on which $\sin\alpha$ does not change sign, and let τ_1 and τ_2 be the two corresponding values of τ. We then readily find that



a) $\varkappa > 0$ b) $\varkappa = 0$ c) $\varkappa < 0$

Fig. 1

$$\tau_2 - \tau_1 = \frac{-\operatorname{sign}\sin\alpha}{\Phi^\circ}\left[\arcsin\frac{\Phi^{\circ 2}\sin\beta_2 - \varkappa\Phi_0}{\cos\varphi\sqrt{\Phi^{\circ 2} - \Phi_0^2}} - \arcsin\frac{\Phi^{\circ 2}\sin\beta_1 - \varkappa\Phi_0}{\cos\varphi\sqrt{\Phi^{\circ 2} - \Phi_0^2}}\right]$$

In view of the symmetry of the kinematic trajectory with respect to the straight lines $\alpha = k\pi$ (k is an integer) and of the fact that the expression following arc sin is equal to ± 1 at the points of intersection of trajectories with these straight lines, we readily obtain the time required for the complete return of the gyroscope frames to their initial position

$$T = \frac{2\pi}{\omega\Phi^\circ} = \frac{2\pi}{\omega\sqrt{1-2a\sin\varphi+a^2}}\,.$$

In the case $a = 0$, this time is, of course, equal to the time $2\pi/\omega$ of one complete revolution of the base.

3. Let us examine the details of the true motion of the gyroscope. To begin with, we note that for $\omega = 0$ problem (1.1) to (1.3) has the unique solution $\alpha \equiv \alpha_0$, $\beta \equiv \beta_0$. Assuming that the solution of this problem for sufficiently small $|\omega|$ can be expanded in series,

$$\alpha(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k(t)\,\omega^k \qquad \beta(t) = \beta_0 + \sum_{k=1}^{\infty} \beta_k(t)\,\omega^k$$

we obtain the following system of linear equations and initial conditions for determining $\alpha_1(t)$, $\beta_1(t)$:

$$f_0\alpha_1^{\cdot\cdot} + (\Omega\cos\beta)\,\beta_1^{\cdot} = -\Omega\Phi_{\alpha_0}', \qquad (-\Omega\cos\beta)\,\alpha_1^{\cdot} + P\beta_1^{\cdot\cdot} = -\Omega\Phi_{\beta_0}' \qquad (3.1)$$

$$\alpha_1(0) = \beta_1(0) = 0, \qquad \alpha_1^{\cdot}(0) = \chi_1, \qquad \beta_1^{\cdot}(0) = \chi_2$$

$$(\Phi_{\alpha_0}' = \Phi_{\alpha}'(\alpha_0, \beta_0) = -\chi_2^*\cos\beta_0, \quad \Phi_{\beta_0}' = \Phi_{\beta}'(\alpha_0, \beta_0) = \chi_1^*\cos\beta_0)$$

In terms of the symbols $M_1 = \sqrt{f_0}\,(\chi_1 - \chi_1^*)$, $M_2 = \sqrt{P}\,(\chi_2 - \chi_2^*)$ the solution of problem (3.1) becomes

$$\alpha_1(t) = \chi_1^* t + \frac{1}{A\sqrt{f_0}}\,[M_1\sin At - M_2(1-\cos At)]$$

$$\beta_1(t) = \chi_2^* t + \frac{1}{A\sqrt{P}}\,[M_2\sin At + M_1(1-\cos At)] \qquad (A = \Omega\cos\beta_0/\sqrt{Pf_0})$$

Here $2\pi/|A|$ is the limit of the period of nutational oscillations of the frames at the point β_0 under the condition that their amplitude tends to zero.

This solution means that the point $\alpha = \alpha_0 + \omega\alpha_1(t)$, $\beta = \beta_0 + \omega\beta_1(t)$ moves in the following way on the plane (α, β): the ellipse with the semiaxes

$$\omega\sqrt{M_1^2 + M_2^2}\,(|A|\sqrt{f_0})^{-1}, \quad \omega\sqrt{M_1^2 + M_2^2}\,(|A|\sqrt{P})^{-1},$$

parallel to the axes α and β moves translationally with the compensating thrust velocity $(\omega\chi_1^*, \omega\chi_2^*)$; at the same time the point (α, β) moves along the ellipse, traversing the entire ellipse in the time $2\pi/|A|$ of a single nutational oscillation. System (3.1) has the first integral

$$f_0\alpha^{\cdot 2} + P\beta^{\cdot 2} = \omega\{2\omega[\Phi_{\alpha_0}'(\alpha_0 - \alpha) + \Phi_{\beta_0}'(\beta_0 - \beta)] + \omega(f_0\chi_1^2 + P\chi_2^2)\}$$

defining the potential well in the form of a half-plane. It is clear that if the velocities $(\alpha^{\cdot}, \beta^{\cdot})$ are parallel at the points $(\alpha^{(1)}, \beta^{(1)})$ and $(\alpha^{(2)}, \beta^{(2)})$ of the well, then the absolute value of the velocity is larger at the point which is further away from the edge of the well.

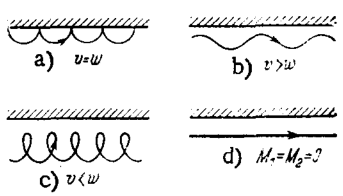


Fig. 2

The velocity of the motion along the ellipse is parallel at two points to the translational velocity of the ellipse; the absolute values of these velocities on the ellipse are equal, but their directions are opposite. At one of these points the velocity of motion along the ellipse is in the same direction as the translational velocity; at the other point it has the opposite direction. In the former point the total velocity is larger in absolute value than at the latter point, so that the first point lies further away from the edge of the potential well than the second.

Fig. 2 shows the possible types of motion. The half-plane complementing the well is shaded. Denoting by v and w the absolute values of the velocities of translational motion of the ellipse and of the motion along the ellipse at the above points, we have $v = w$ in case (a), $v > w$ in case (b), and $v < w$ in case (c) ; in all three cases we assume that $M_1^2 + M_2^2 > 0$, case (d) shows the motion for $M_1 = M_2 = 0$ (i.e. the case where a compensating thrust is present).

Thus, during one period $T_1 = 2\pi / |A|$ of nutational oscillation the point (α, β) moves from its initial position (α_0, β_0) to the point $(\alpha_0 + \omega\chi_1^* T_1, \beta_0 + \omega\chi_2^* T_1)$, i.e. in the direction of the tangent to the kinematic trajectory. Assuming that $|\mu| = \omega / |\Omega|$ is very small, we can conclude that the point has simply moved along the kinematic trajectory to some point α_0°, β_0°, etc. Thus, if we neglect the nutational oscillation, we can assume that the point (α, β) moves discretely along the kinematic trajectory, with the average velocity of each jump equal to the velocity of the compensating thrust at the corresponding point of the trajectory. This type of motion can readily be discribed by means of difference equations. If instead of the difference equations we write out their continuous analog, we arrive at problem (1.9) which defines the kinematic trajectory. From now on we shall always assume that the true trajectory for problem (1.1) to (1.3) is of the form shown in Fig. 3, where the domain complementing the potential well is shaded.

4. Let (α_0, β_0) be such that the corresponding kinematic trajectory $\Phi(\alpha, \beta) = \Phi_0$ is far from the fixed points of the function $\Phi(\alpha, \beta)$. Then for sufficiently small $|\mu|$ some strip surrounding the kinematic trajectory contains no fixed points of the function $F(\alpha, \beta)$. Through each point of this strip passes a line $F =$ const for which the tangent vector $(\Omega^{-1}F_\beta', -\Omega^{-1}F_\alpha')$ forms an acute angle with the direction of the kinematic trajectory.

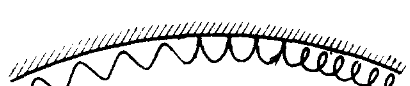


Fig. 3

Let $\alpha(t)$, $\beta(t)$ be a solution of problem (1.1) to (1.3) and let the function $\delta(t) = F(\alpha(t), \beta(t))$ reach a relative extremum for some t_0 . The point $(\alpha(t_0), \beta(t_0))$ will be called a marked point of the first kind if this extremum is a minimum, and a marked point of the second kind if it is a maximum. If a point of the trajectory lies on the edge of the well, then it is a marked point of the second kind, and the trajectory has a cusp there. From the considerations presented in connection with model problem (3.1) we conclude that at the marked points of the first kind, the velocity vector $(\alpha^\cdot(t_0), \beta^\cdot(t_0))$ has the same direction as the tangent to the line $F =$ const . At the marked points of the second kind not lying on the edge of the well, the indicated directions can be either the same or opposite (see Fig. 3).

Let the interior point (α, β) of the well be a marked point of the first kind of the true trajectory. Then, on the one hand, we have Eq. (1.4), while on the other hand

$$\alpha^\cdot = k\Omega^{-1} F_\beta', \quad \beta^\cdot = -k\Omega^{-1} F_\alpha' \qquad (k > 0)$$

Let us introduce the notation

$$H(\alpha, \beta) = f(\beta)(F_\beta')^2 + P(F_\alpha')^2$$

$$B = \left(\frac{2\omega[F_0 - F(\alpha, \beta)] + \omega^2(f_0\chi_1^2 + P\chi_2^2)}{H}\right)^{1/2} \qquad (\eta = \operatorname{sign}\Omega)$$

Then

$$\alpha^\cdot = \eta B F_\beta', \qquad \beta^\cdot = -\eta B F_\alpha' \tag{4.1}$$

Further, we have

$$\frac{dF}{dt} = F_\alpha'\alpha^\cdot + F_\beta'\beta^\cdot, \qquad \frac{d^2F}{dt^2} = F_{\alpha\alpha}''\alpha^{\cdot 2} + 2F_{\alpha\beta}''\alpha^\cdot\beta^\cdot + F_{\beta\beta}'\beta^{\cdot 2} + F_\alpha'\alpha^{\cdot\cdot} + F_\beta'\beta^{\cdot\cdot}$$

Replacing $\alpha^{\cdot\cdot}$, $\beta^{\cdot\cdot}$ by their expressions obtained from (1.1) and (1.2), respectively, and recalling Formulas (4.1), we find that at the marked points of the first kind we have Eq.

$$2Pf(\beta)\frac{d^2F}{dt^2} = -2\omega H + 2\eta(\Omega\cos\beta + \omega p)HB +$$

$$+ B^2\{2Pf(\beta)[F_{\alpha\alpha}''(F_\beta')^2 - 2F_{\alpha\beta}''F_\alpha'F_\beta' + F_{\beta\beta}''(F_\alpha')^2] + f'(\beta)F_\beta'[2P(F_\beta')^2 + f(\beta)(F_\beta')^2]\}$$

The right-hand side in this expression is a known function of α and β. Let us denote it by $K(\alpha,\beta)$. Since $d^2F/dt^2 \geq 0$ at the marked points of the first kind, we have $K(\alpha,\beta) \geq 0$ at these points. On the other hand, by virtue of the fact that $B=0$ on the edge of the potential well we have $K = -2\omega H < 0$; since $K(\alpha,\beta)$ is continuous, this inequality is also valid in some strip at the well edge, so that this strip contains no marked points of the first kind. Clearly, as $\omega \to 0$

$$\frac{K(\alpha,\beta)}{\sqrt{\omega}} \to 2\cos\beta\sqrt{\frac{2\Omega[\Phi_0 - \Phi(\alpha,\beta)]}{f(\beta)(\Phi_\beta')^2 + P(\Phi_\alpha')^2}} > 0$$

at each interior point of the domain G_0, so that for sufficiently small ω the domain G_ω contains points for which $K > 0$. Thus, the line $K = 0$ separates the marked points of the first kind from the well edge. If the vector $(\alpha^\cdot, \beta^\cdot)$ has the same direction as the vector $(\Omega^{-1}F_\beta', -\Omega^{-1}F_\alpha')$, at all the marked points of the second kind in some portion of the true trajectory, it is necessarily the case that $K \leq 0$ at these points, so that the line $K = 0$ intersects the true trajectory several times and is a good approximation to it. The line $K = 0$ will be called the "energy trajectory" inasmuch as it takes account of the energy of the initial thrust in the form $f_0\chi_1^2 + P\chi_2^2$.

The shortcomings of this trajectory include the following: (1) we do not know the "schedule" of motion along this trajectory, since it is not defined by equations of the (1.9) type; (2) generally speaking, this trajectory does not pass through the point (α_0, β_0); (3) if the vector $(\alpha^\cdot, \beta^\cdot)$ is directed oppositely to the vector $(\Omega^{-1}F_\beta', -\Omega^{-1}F_\alpha')$ at the marked points of the second kind in some portion of the true trajectory, then it may turn out that $K > 0$ at these marked points, so that this portion of the true trajectory lies on one side of the enrgy trajectory and thereby separates the true trajectory from the well edge. For example, in the case of model problem (3.1) the marked points of the first and second kind lie on different sides of the energy trajectory if and only if

$$\sqrt{M_1^2 + M_2^2} \leqslant 2\sqrt{f_0\chi_1^{*2} + P\chi_2^{*2}}$$

i.e. $|\chi_1 - \chi_1^*|$, $|\chi_2 - \chi_2^*|$ are not large as compated to $|\chi_1^*|$, $|\chi_2^*|$.

We note that $K = 0$ for all the points of the kinematic trajectory $\Phi(\alpha,\beta) = \Phi_0$ if all of the following conditions are fulfilled: (a) $P = Q = R$, $S = 0$ (e.g. in the case of inertialess frames), (b) $a = 0$ (for an undisplaced rotor), (c) $\chi_1 = \chi_1^*$, $\chi_2 = \chi_2^*$ (with a compensating thrust). In the general case, considering Eq. $K = 0$ as quadratic in B, we obtain one of the roots B_1 which is positive and tends to zero as $\omega \to 0$. Hence, the equation of energy trajectory can also be written as

$$\Phi(\alpha,\beta) - \Phi_0 = -HB_1^2/2\omega\Omega + \mu(\Psi_0 - \Psi(\alpha,\beta)) + {}^1/_2\mu(f_0\chi_1^2 + P\chi_2^2)$$

where the expansion of the first term of the right-hand side in a power series can be written as

$$\frac{HB_1^2}{2\omega\Omega} = \frac{\mu}{2}\left[f(\beta)\left(\frac{\Phi_\beta'}{\cos\beta}\right)^2 + P\left(\frac{\Phi_\alpha'}{\cos\beta}\right)^2\right] + \ldots$$

This implies that Formula (1.8) in which $d\alpha/d\tau$ and $d\beta/d\tau$ are replaced by the right-hand sides of (1.9) can be regarded as an approximate equation (to within μ^2) of the energy trajectory.

Assuming that the energy trajectory is very close to the true trajectory, we can estimate the error of gyroscope operation in the case $a = 0$. To do this we replace the left-hand side of the approximate equation just written by $\cos\gamma_0 - \cos\gamma$ and α, β in its right-hand side by their values for the kinematic trajectory. This yields an analog of Formula (1.10) which results from adding the term $[f_0(\chi_1^2 - \chi_1^{*2}) + P(\chi_2^2 - \chi_2^{*2})]$ to the expression in braces in Formula (1.10).

5. Merely to be specific we shall carry out all of our subsequent analysis for the case $a = 0$, $\varkappa = -\sin\varphi > 0$, $|\Phi(\alpha_0, \beta_0)| < \varkappa$ (the kinematic trajectory is open and α increases).

The kinematic, potential, and energy trajectories are symmetrical with respect to the straight lines $\alpha = k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$. We can prove that the true trajectory is symmetrical with respect to the straight line $\alpha = j\pi$ (j is an integer) provided that $\beta^{\cdot} = 0$ at the instant of intersection of this straight line by the trajectroy; this point of the trajectory is then marked. It is clear that by a slight alteration of the initial data we can guarantee that a trajectory close to the one under consideration has a marked point on the straight line $\alpha = j\pi$. In this sense the true trajectory can be considered approximately symmetrical with respect to the straight lines $\alpha = j\pi$. Hence, if, for example, $0 < \alpha_0 < \pi$, then we must first find the segment of the trajectory $\alpha_0 \leq \alpha \leq \pi$ and reflect it symmetrically with respect to the straight line $\alpha = \pi$ onto the segment $\pi \leq \alpha \leq 2\pi - \alpha_0$, then find independently the segment of the trajectory for $2\pi - \alpha_0 \leq \alpha \leq 2\pi$ from the initial data $\alpha = 2\pi - \alpha_0$, $\beta = \beta_0$, $\alpha^{\cdot} = -\alpha^{\cdot}_0 = -\omega\chi_1$, $\beta^{\cdot} = \beta^{\cdot}_0 = \omega\chi_2$ and continue reflection further on in the straight line $\alpha = 2\pi$, etc.

Let (α_k, β_k) and t_k $(k = 1, 2, 3, \ldots)$ be the sequences of all marked points of the first kind and of the times of passage through them for some true trajectory. Let the point with the subscript N be closest to $(\alpha_0 + 2\pi, \beta_0)$. Then t_N is, of course, the time of complete return of the gyroscope to its initial position. The broken line connecting in sequence the points (α_k, β_k), will be called a marked trajectory of the first kind. Similarly, we can consider a marked trajectory of the second kind as one which connects points of the second kind. In fact, the trajectory lies in the strip between the marked trajectories of the first and second kind. In the time $2\pi/\omega$ of one complete revolution of the base it forms approximately $2\pi/10\omega$ nutational loops in this strip (if we assume that $\cos\beta > \frac{1}{2}$ on the trajectory and that $Pf(\beta)$ is a quantity of order from 1 to 5). If the rotor executes 1000 revolutions per second and that the gyroscope is situated on the Earth's surface, then the number of nutational oscillations is approximately 9×10^6. In solving problem (1.1) to (1.3) by some numerical method (Runge-Kutta, Adams-Störmer, etc.) it is advisable to choose the time interval in such a way that several intervals (from 10 to 40) fit within a single nutational oscillation. If we solve the problem for a semitrajectory and take just 10 intervals per nutational oscillation, this means 45×10^6 intervals. It appears that the solution of such a problem on existing computers is unfeasible because of the machine time required and the unreliability of the final results.

The subject of the present paper was suggested by a discussion with N. V. Butenin of the possibility of solving problem (1.1) to (1.3) on a computer. Paper [5] contains some results obtained through the numerical solution of the problem in question.

Thus, determination of the complete true trajectory is a time-consuming and apparently unnecessary problem. We shall now describe a method for the approximate determination of the strip between the marked trajectories.

Let us denote G_ω^+ the domain in which the inequality $K > 0$ is fulfilled. If the point (α_0, β_0) lies in G_ω^+ and if $\alpha_0^{\cdot} = \omega\chi_1$, $\beta_0^{\cdot} = \omega\chi_2$ satisfy Eqs. (4.1) for $\alpha = \alpha_0$, $\beta = \beta_0$, then the initial point (α_0, β_0) is the first marked point of the first kind in the required trajectory. If (α_0, β_0) is not such a point, then by some method of numerical integration of problem (1.1) to (1.3) (with an interval substantially smaller than the nutational oscillation period) we can find the first marked point of the first kind on the basis of a change in sign of the quantity $F_\alpha'\alpha^{\cdot} + F_\beta'\beta^{\cdot})$ from minus to plus.

Now let us assume that we have found some point $(\alpha^\circ, \beta^\circ)$ of the domain G_ω^+ by some method and have established that it is a marked point of the first kind in the required trajectory, or that it lies close to the latter. Then, setting $\alpha = \alpha^\circ$, $\beta = \beta^\circ$ in Formula (4.1), we find $\alpha^{\circ\cdot}$, $\beta^{\circ\cdot}$ and carry out the numerical integration of system (1.1), (1.2) under the initial conditions $\alpha^\circ, \beta^\circ, \alpha^{\circ\cdot}, \beta^{\circ\cdot}$ to obtain a segment of the trajectory extending over 5 to 10 nutational oscillations. The memory need only store the coordinates of the marked points of the first and second kind, the times of passage through these points, and the directions of the velocities at the marked points of the second kind. The coordinates and times of passage through the points of the first kind are treated (e.g. by the method of least squares) in order to find the vector (r, s) of the average velocity of the marked point of the first kind. This vector (r, s) and the time t_1 (which spans 100 to 200 periods of nutational oscillation) can be used to find the point $\alpha^{\circ\circ} = \alpha^\circ + rt_1$, $\beta^{\circ\circ} = \beta^\circ + st_1$. If the point $(\alpha^{\circ\circ}, \beta^{\circ\circ})$ belongs to the domain G_ω^+, then $(\alpha^{\circ\circ}, \beta^{\circ\circ})$ is taken as the new approximate marked point of the first kind in the required trajectory. If $(\alpha^{\circ\circ}, \beta^{\circ\circ})$ does not belong to the domain G_ω^+, then the domain G_ω^+ and the point $(\alpha^\circ + {}^1/_2\, rt_1, \beta^\circ + {}^1/_2\, st_1)$ are also checked to see whether they belong, etc. Once a point from the domain G_ω^+ has been found in this way, the step of determining the average velocity of the point of the first kind and the extrapolation are repeated. The coordinates and directions of the velocity at the marked points of the second kind which are stored in the memory make it possible to estimate the width of the strip occupied by the trajectory and provide a notion of the character of the nutational loops.

No actual computations were carried out in the course of the present study. Experimental computations of the complete true trajectory and of the strip between the marked trajectories should probably be carried out using not excessively small values (on the order of 0.0001 to 0,01).

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APPLICATION OF THE LIAPUNOV METHOD TO LINEAR SYSTEMS WITH LAG

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V. B. KOLMANOVSKII
(Moscow)
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Several authors have investigated the properties of the solutions of differential equations of the form

$$x^{\cdot}(t) = -\int_0^\infty x(t-s)\,dK_0(t,s) \tag{1}$$

$$x^{\cdot\cdot}(t) = -\int_0^\infty x^{\cdot}(t-s)\,dK_1(t,s) - \int_0^\infty x(t-s)\,dK_2(t,s) \tag{2}$$

Here and below the symbol $dK_i(t,s)$ denotes the differential with respect to the second argument. Attention has been largely confined to the case of concentrated lag, i.e. to step functions $K_i(t,s)$. The general case of lag distributed over a finite interval $[0, s(t)]$ was first investigated by Myshkis [1].

Stability conditions for the solutions of differential equations of this type were obtained in [2] under the assumption that the kernels K_i depend only on s, i.e. that $K_i(t,s) \equiv K_i(s)$ and that the variation of the functions $K_i(s)$ in $[0, \infty)$ is bounded.

The present paper concerns the stability conditions for trivial solutions of equations of the form (1) and (2).

We assume that the functions $K_i(t,s)$ satisfy the following requirement of bounded variation with respect to s:

$$\sup_t \int_0^\infty |dK_i(t,s)| \leqslant \text{const} < \infty \qquad (0 \leqslant t < \infty) \tag{3}$$

The solution $x(t)$ of Eq. (1) (Eq. (2)) for $t>0$ is determined by the function $\varphi(t)$ (the function $\psi(t)$) specified on $(-\infty, 0]$,

$$x(t) = \varphi(t), \qquad t \leqslant 0 \tag{4}$$

$$(x(t) = \psi(t),\ x^{\cdot}(t) = \psi^{\cdot}(t),\ t \leqslant 0) \tag{5}$$

Definition. The solution $x_1(t)$ of problem (1), (4) (the solution $x_2(t)$ of problem (2), (5)) will be called "stable" if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$|x_1(t)| < \varepsilon \quad \text{for } t \geqslant 0 \qquad (|x_2(t)| < \varepsilon \quad \text{for } t \geqslant 0)$$